

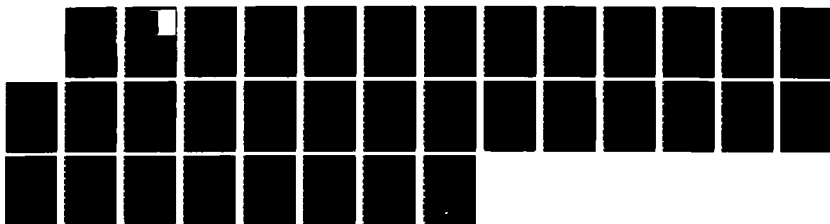
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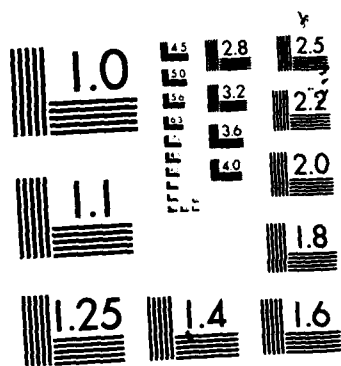
AN $L(P)$ -THEORY FOR THE N-DIMENSIONAL STATIONARY
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STATIONARY, COMPRESSIBLE, NAVIER-STOKES
EQUATIONS, AND THE INCOMPRESSIBLE
LIMIT FOR COMPRESSIBLE FLUIDS.
THE EQUILIBRIUM SOLUTIONS

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ABSTRACT

In this paper we study the system (1.1), (1.3), which describes the stationary motion of a given amount of a compressible heat conducting, viscous fluid in a bounded domain Ω of \mathbb{R}^n , $n > 2$. Here $u(x)$ is the velocity field, $\rho(x)$ is the density of the fluid, $\zeta(x)$ is the absolute temperature, $f(x)$ and $h(x)$ are the assigned external force field and heat sources per unit mass, and $p(\rho, \zeta)$ is the pressure. In the physically significant case one has $g = 0$. We prove that for small data (f, g, h) there exists a unique solution (u, ρ, ζ) of problem (1.1), (1.3), in a neighborhood of $(0, m, \zeta_0)$; for arbitrarily large data the stationary solution does not exist in general. (see section 5). Moreover, we prove that (for barotropic flows) the stationary solution of the Navier-Stokes equations (4.15) is the incompressible limit of the stationary solutions of the compressible Navier-Stokes equations (4.14), as the Mach number becomes small. Finally, in section 5 we will study the equilibrium solutions for system (4.1). For a more detailed explanation see the introduction.

AMS (MOS) Subject Classifications: 35G30, 35M05, 35Q10, 76D05, 76N10.

Key Words: Non-linear partial differential equations, viscous compressible fluids, Incompressible limit, Navier-Stokes equations.

Work Unit Number 1 - Applied Analysis

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AN L^p -THEORY FOR THE n -DIMENSIONAL, STATIONARY, COMPRESSIBLE NAVIER-STOKES EQUATIONS,
AND THE INCOMPRESSIBLE LIMIT FOR COMPRESSIBLE FLUIDS. THE EQUILIBRIUM SOLUTIONS

H. Beirão da Veiga*

1. Introduction

In this paper we study the system

$$(1.1) \quad \begin{cases} -\mu \Delta u - \nu \nabla \operatorname{div} u + \nabla p(\rho, \zeta) = \rho [f - (u \cdot \nabla)u] , \\ \operatorname{div}(\rho u) = g, \\ -\chi \Delta \zeta + c_v \rho u \cdot \nabla \zeta + \zeta p'_\zeta(\rho, \zeta) \operatorname{div} u = \rho h + \psi(u, u), \text{ in } \Omega , \\ u|_\Gamma = 0, \zeta|_\Gamma = \zeta_0 , \end{cases}$$

in a bounded open domain Ω in \mathbb{R}^n , for arbitrarily large $n \geq 2$. It is assumed that Ω lies (locally) on one side of its boundary Γ , a C^2 manifold. Here,

$$(1.2) \quad \psi(u, u) = \chi_0 \sum_{i,j=1}^n \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 + \chi_1 (\operatorname{div} u)^2 ,$$

and $(v \cdot \nabla)u = \sum_{i=1}^n v_i (\partial u / \partial x_i)$. System (1.1) describes the stationary motion of a compressible, heat-conductive, viscous fluid (see Serrin [10]). In equation (1.1), $u(x) = (u_1(x), u_2(x), \dots, u_n(x))$ is the velocity field, $\rho(x)$ is the density of the fluid, $\zeta(x)$ is the absolute temperature, $f(x)$ and $h(x)$ are the assigned external force field and heat sources per unit mass, and $p(\rho, \zeta)$ is the pressure. In the physically significant case one has $g = 0$, however, it is not without interest, from a mathematical point of view, to study the general case.

In order to avoid technicalities, we will assume that the coefficients $\mu > 0$, $\nu > -\mu$, $\chi > 0$, c_v , χ_0 , χ_1 , are constant. A dependence of those coefficients on u, ρ, ζ , as well as the introduction in equation (1.1) of other kinds of non-linearities, does not give rise to substantial difficulties. For the same reason, we assume that $\zeta_0 > 0$ is constant.

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Since the total mass of fluid is given, we impose the condition

$$(1.3) \quad \frac{1}{|\Omega|} \int_{\Omega} \rho(x) dx = m, \text{ or equivalently } \frac{1}{|\Omega|} \int_{\Omega} \sigma(x) dx = 0,$$

where $m > 0$ is given, and $\sigma(x)$ is defined by setting $\rho(x) \equiv m + \sigma(x)$.

The function $p(\rho, \zeta)$ is defined, and has Lipschitz continuous first order partial derivatives p'_ρ and p'_ζ in a neighborhood $[m-l, m+l] \times [\zeta_0-l_1, \zeta_0+l_1]$ of (m, ζ_0) , where $0 < l < m/2$, $0 < l_1 < \zeta_0/2$. Consequently, we can write

$$(1.4) \quad \begin{cases} p'_\rho(m+\sigma, \zeta_0+\alpha) = k + \omega_1(\sigma, \alpha) \\ p'_\zeta(m+\sigma, \zeta_0+\alpha) = \omega_2(\sigma, \alpha) \end{cases},$$

where $k = p'_\rho(m, \zeta_0)$, $\omega_1(0, 0) = 0$, and ω_1, ω_2 are Lipschitz continuous in $I(l, l_1) \equiv [-l, l] \times [-l_1, l_1]$. We assume that $k > 0$ (in fact, $k \neq 0$ would be sufficient here).

By setting

$$\rho = m + \sigma, \quad \zeta = \zeta_0 + \alpha,$$

we write system (1.1) in the equivalent form

$$(1.5) \quad \begin{cases} -\mu \Delta u - \nu \nabla \operatorname{div} u + k \nabla \sigma = F(f, u, \sigma, \alpha) \\ m \operatorname{div} u + u \cdot \nabla \sigma + \sigma \operatorname{div} u = g \\ -\chi \Delta \alpha = H(h, u, \sigma, \alpha), \text{ in } \Omega \\ u|_{\Gamma} = 0, \alpha|_{\Gamma} = 0 \end{cases},$$

where, by definition,

$$(1.6) \quad \begin{cases} F(f, u, \sigma, \alpha) = (\sigma + m)[f - (u \cdot \nabla)u] - \omega_1(\sigma, \alpha) \nabla \sigma - \omega_2(\sigma, \alpha) \nabla \alpha \\ H(h, u, \sigma, \alpha) = (\sigma + m)h - c_v(m + \sigma)u \cdot \nabla \alpha + \psi(u, u) + \frac{\zeta_0 + \alpha}{m + \sigma} \omega_2(\sigma, \alpha)(u \cdot \nabla \sigma - g) \end{cases}.$$

Note that equation (1.5)₂ is used in deducing the expression H .

Let $p > n$ be fixed. We prove that problem (1.1), (1.3) has a unique solution $(u, \rho, \zeta) \in W^{2,p} \times W^{1,p} \times W^{2,p}$ in a neighborhood of $(0, m, \zeta_0)$, provided the data $(f, g, h) \in L^p \times \overline{W}_0^{1,p} \times L^p$ belong to a suitable neighborhood of the origin. See theorem 3.1. This result can be generalized as follows:

Theorem 1.1. Let $p \in]1, +\infty[$ and $j > -1$, verify $j+2 > n/p$. Assume that $\Gamma \in C^{3+j}$ and

$p(\rho, \zeta) \in C^{3+j}(I(l, l_1))$. If $(f, g, h) \in W_0^{j+1, p} \times W_0^{j+2, p} \times W_0^{j+1, p}$ verify the condition

$$\|f\|_{j+1, p} + \|g\|_{j+2, p} + \|h\|_{j+1, p} < c'_0,$$

then there exists a unique solution $(u, \rho, \zeta) \in W_0^{j+3, p} \times W_0^{j+2, p} \times W_0^{j+3, p}$ of problem (1.1),

(1.3) in the ball

$$\|u\|_{j+3, p} + \|\rho - m\|_{j+2, p} + \|\zeta - \zeta_0\|_{j+3, p} < c'_1.$$

In the above statement c'_0, c'_1 are suitable positive constants depending only on $\Omega, n, p, j, \mu, \nu, k, m, \zeta_0, c_v, X, X_0, l, l_1, T_1, S_1 (i=1, 2)$, where $T_1 = \sup |\omega_1(\sigma, \alpha)|$ for $(\sigma, \alpha) \in I(l, l_1)$, and S_1 is the norm of $\omega_1(\sigma, \alpha)$ in the space $C^{2+j}(I(l, l_1))$.

The LP-theory enables us to treat the n -dimensional case by handling only derivatives of order less than or equal to two (case $j = -1$). Nevertheless, our proof applies as well to the case $j > -1$, without further difficulties or technical calculations, as shown in the last part of section 3; the details will be given in a forthcoming note, by

A. Defranceschi. To be more specific, here we will concentrate our attention on the case $j = -1$, since our proof turns out to be (slightly) more complicated just for this case.

Section 4 is concerned with barotropic motions, described by system (4.1). The main goal of this section is the study of the incompressible limit of a family of compressible barotropic flows. We assume that the state function $p_\lambda(\rho)$ depends on a parameter λ , such that the Mach number becomes small as λ goes to $+\infty$. We refer to [5], [6] for the physical justification of our assumptions. We will prove that the solution of the incompressible Navier-Stokes equation (4.15) is the limit of the solutions of the compressible Navier-Stokes system (4.14), as $\lambda \rightarrow +\infty$. A similar result was proved for $n < 3, j = 1, p = 2$, in reference [2]. The proof given here applies, without supplemental difficulties, to the case $W_0^{j, p}$, if $j+2 > n/p, \Gamma \in C^{3+j}, p(\rho) \in C^{3+j}$. The details will be given in the forthcoming note, referred to above.

Finally, in section 5 we will study the existence of equilibrium solutions (solutions such that $u(x) = 0, \forall x \in \Omega$), for arbitrarily large external forces $f = \nabla F$. In particular, we will show that if $p(\rho) = R\rho^\gamma, R > 0, \gamma > 0, \gamma \neq 1$, the equilibrium solution does not exist in general (even for a constant external force f). If $\gamma = 1$,

then the equilibrium solution exists for every bounded potential $F(x)$.

For small external forces, it was proved in [11] that the stationary solutions are stable (for the equilibrium solutions this was proved in [7]). It would be interesting to study the stability of the equilibrium solutions for large external forces, even if $n = 1$.

It is worth noting that the core of this paper is the study of the linear system (2.1); see theorems 2.1 and 3.3. In this system the unknowns are u and σ , the vector field v being fixed. Due to the term $v \cdot \nabla \sigma$, system (2.1) is not an elliptic system in the sense of Agmon, Douglis, and Nirenberg [1; part II], except if $v(x)$ vanishes identically in Ω . In fact (assume, for simplicity, that $\mu = k = m = 1$, and $v = 0$), consider in Ω the system $-\Delta u + \nabla \sigma = F$, $\operatorname{div} u + v \cdot \nabla \sigma + a(x)\sigma = g$. By using the notations of [1; part II] one has: $L(x, \xi) = v(x) \cdot \xi |\xi|^{2n}$ if $v(x) \neq 0$; $L(x, \xi) = (1+a(x)) |\xi|^{2n}$ if $v(x) = 0$. Consequently, the ellipticity condition (see [1], eq. (1.5)) " $L(x, \xi) \neq 0$ for real $\xi \neq 0$ ", cannot be satisfied unless $v(x) = 0$, for all $x \in \Omega$.

If $v = 0$ and if a is small enough (for instance, if $|a|_\infty < 1$), the system is elliptic. However, this last property cannot be used to treat the term $v \cdot \nabla \sigma$ as a perturbation term. In fact, by assuming that $\sigma \in W^{1,p}$, the term $v \cdot \nabla \sigma$ belongs just to L^p . In this situation, equation $\operatorname{div} u + v \cdot \nabla \sigma + a(x)\sigma = g$ yields $\operatorname{div} u \in L^p$, and equation $-\Delta u + \nabla \sigma = F$ gives $u \in W^{1,p}$, $\sigma \in L^p$. Hence, from the point of view of regularity, we lose one derivative.

We also point out that the evolution problem is easier to solve due to the presence of the term $(\partial \sigma / \partial t) + v \cdot \nabla \sigma$ in the evolution counterpart of equation (2.1)₂.

Let us briefly explain the main ideas utilized in the sequel (see also [2]) to solve the linear system (2.1): By applying the divergence operator to both sides of equation (2.1)₁, one shows that $(\mu + v) \Delta \operatorname{div} u = k \Delta \sigma - \operatorname{div} F$, and by applying the Laplace operator to both sides of equation (2.1)₂, one gets

$$(1.7) \quad \frac{mk}{\mu + v} \Delta \sigma + v \cdot \nabla \Delta \sigma = G(F, g, \sigma),$$

where $G(F, g, \sigma)$ is given by (2.5). In order to be able to solve equation (1.7) for $\Delta \sigma$, we replace $G(F, g, \sigma)$ by $G(F, g, \tau)$, where $\tau \in \bar{W}^{1,p}$ is an arbitrary function. This yields equation (2.4), where (heuristically) λ should be regarded as $\Delta \sigma$ (actually

$\lambda = \Delta\sigma$, if $\tau = \sigma$ is a solution of (2.1)). Once λ is known, we want to determine $\text{div } u$. By applying the divergence operator to both sides of (2.1)₁, we get equation (2.10)₁, where (heuristically) λ should be regarded as $\text{div } u$ (we will show that $\theta = \text{div } u$, in the event that we have a fixed point $\sigma = \tau$). The remarkable property $\text{div } u = 0$ on Γ ⁽¹⁾ suggest us to impose on θ the boundary condition (2.10)₂. Once θ is known, equation (2.1)₁ gives u and σ , by solving the Stokes linear problem (2.13). In equation (2.13) we replace $\theta(x)$ by $\theta_0(x) = \theta(x) - \bar{\theta}$, where $\bar{\theta}$ is the mean value of $\theta(x)$, since the compatibility condition $\bar{\theta}_0 = 0$ is required here.

For each fixed pair (F, g) the above sequence of maps define a map $\tau \rightarrow \sigma$. We will prove that this map is a contraction (in a suitable ball), and that the pair (u, σ) , corresponding to the fixed point $\sigma = \tau$, is a solution of (2.11).

We end this section by calling the reader's attention to the following papers, in which results directly related with those of section 3 can be found: M. Padula [8], A. Valli [11], A. Valli and W. Zajackowski [13], M. Beirão da Veiga [2], A. Valli [12].

The corresponding evolution problem has been studied by many authors. Here, we mention only a sequence of papers by Matsumura and Nishida (see [7], and references) and the interesting Valli's paper [11].

To readers interested in the incompressible limit of compressible fluids, we suggest papers by Klainerman and Majda [5], Majda [6], and Schochet [9].

⁽¹⁾ Equation (1.5)₂ implies that every stationary solution u must verify the equation $\text{div } u = 0$ on Γ .

2. The linearized system

In this section we study the linear system

$$\begin{aligned} (2.1) \quad & -\mu \Delta u - \nu \nabla \operatorname{div} u + k \nabla \sigma = F(x), \\ & \mu \operatorname{div} u + \nu \nabla \sigma + \sigma \operatorname{div} \nu = g, \quad \text{in } \Omega, \\ & u|_{\Gamma} = 0, \end{aligned}$$

by adapting to the L^p -case the method introduced in [2]. We start with some notation. We denote by $W^{j,p}$, j an integer, $1 < p < +\infty$, the Sobolev space $W^{j,p}(\Omega)$, endowed with the usual norm $\|\cdot\|_{j,p}$, and by $\|\cdot\|_p$, $1 < p < +\infty$, the usual norm in $L^p = L^p(\Omega)$. Hence, $\|\cdot\|_{0,p} = \|\cdot\|_p$. For convenience, we also use the symbol $W^{j,p}$ to denote the space of vector fields v in Ω such that $v_i \in W^{j,p}(\Omega)$, $i = 1, 2, \dots, n$. This convention applies to all the functional spaces and norms utilized here. For $j > 1$ we define $W_0^{j,p} = \{v \in W^{j,p} : v = 0 \text{ on } \Gamma\}$. Note that $W_0^{j,p} = W^{j,p} \cap W_0^{1,p}$, is not the closure of $D(\Omega) = C_0^\infty(\Omega)$. Furthermore, we set $\bar{W}^{j,p} = \{v \in W^{j,p} : \bar{v} = 0\}$, $\bar{W}_0^{j,p} = W_0^{j,p} \cap \bar{W}^{j,p}$, $j > 1$, where in general $\bar{\phi}$ denotes the mean value of $\phi(x)$ in Ω . Finally, for vector fields, we define $W_{0,d}^{j,p} = \{v \in W_0^{j,p} : \operatorname{div} v = 0 \text{ on } \Gamma\}$, $j > 2$.

With the only exception for section 5, c, c_i , $i > 0$, will denote positive constants depending at most on Ω, n, p . The symbol c may be utilized (even in the same equation) to indicate distinct constants. In section 5, c will denote an integration constant.

We denote by $q = p/(p-1)$ the dual exponent of p , and by r^* the Sobolev embedding exponent $r^* = (n-r)/nr$. Recall that $W^{1,p} \hookrightarrow L^\infty$, if $p > n$, and $W^{1,p} \hookrightarrow L^{p^*}$, if $1 < p < n$.

Now we state the main result in this section (see also theorem 3.3):

THEOREM 2.1. Let $p > n$ be fixed, and let $F \in L^p$, $g \in \bar{W}_0^{1,p}$. There exist positive constants c and γ, γ defined by equation (2.22), such that if $v \in W_{0,d}^{2,p}$ verifies the assumption

$$(2.2) \quad \|v\|_{2,p} < \gamma k,$$

then there exists a unique solution $(u, \sigma) \in W_{0,d}^{2,p} \times \bar{W}^{1,p}$ of problem (2.1). Moreover,

$$(2.3) \quad \mu \|u\|_{2,p} + k \|\sigma\|_{1,p} < c \left(1 + \frac{\mu + |v|}{\mu + \nu}\right) \|F\|_p + c \frac{\mu + |v|}{m} \|g\|_{1,p}.$$

Proof. Let $\tau \in \bar{W}^{-1,p}$, and consider the linear problem

$$(2.4) \quad \frac{mk}{\mu+\nu} \lambda + v \cdot \nabla \lambda = G(F, g, \tau),$$

where by definition

$$(2.5) \quad G(F, g, \tau) = \Delta g + \frac{m}{\mu+\nu} \operatorname{div} F - [2\nabla v : \nabla^2 \tau + \Delta v \cdot \nabla \tau + \Delta(\tau \operatorname{div} v)].$$

Here, $\nabla v : \nabla^2 \tau = \sum (\partial v_i / \partial x_k) (\partial^2 \tau / \partial x_i \partial x_k)$. Note that G is a linear map from $L^p \times \bar{W}_0^{-1,p} \times \bar{W}^{-1,p}$ into $W^{-1,p}$ (the dual space of $W_0^{1,q}$), and that

$$(2.6) \quad \|G\|_{-1,p} \leq \|g\|_{1,p} + \frac{m}{\mu+\nu} \|F\|_p + c \|v\|_{2,p} \|\tau\|_{1,p}.$$

In [3] we prove that there exists a positive constant c_1 such that if

$\|v\|_{2,p} \leq c_1 mk/(\mu+\nu)$, then there exists a linear continuous map $L : W^{-1,p} \rightarrow W^{-1,p}$ such that $\lambda = LG$ is a weak solution of equation (2.4), for every $G \in W^{-1,p}$. Moreover

$$(2.7) \quad \frac{mk}{\mu+\nu} \|\lambda\|_{-1,p} \leq c \|G\|_{-1,p}.$$

By a weak solution of (2.4) (see [3] for details), we mean a distribution $\lambda \in W^{-1,p}$ such that

$$(2.8) \quad \left\langle \frac{mk}{\mu+\nu} \varphi - \operatorname{div}(\varphi v), \lambda \right\rangle = \langle \varphi, G \rangle, \quad \forall \varphi \in (\Omega).$$

Here, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W_0^{1,q}$ and $W^{-1,p}$. The above result yields

$$(2.9) \quad \frac{mk}{\mu+\nu} \|\lambda\|_{-1,p} \leq c \left(\frac{m}{\mu+\nu} \|F\|_p + \|g\|_{1,p} + \|v\|_{2,p} \|\tau\|_{1,p} \right).$$

Now let $\theta \in W_0^{1,p}$ be the solution of the Dirichlet problem

$$(2.10) \quad \begin{cases} (\mu+\nu)\Delta\theta = k\lambda - \operatorname{div} F & \text{in } \Omega, \\ \theta|_{\Gamma} = 0. \end{cases}$$

Clearly,

$$(2.11) \quad (\mu+\nu)\|\theta\|_{1,p} \leq c \|F\|_p + c \frac{\mu+\nu}{m} (\|g\|_{1,p} + \|v\|_{2,p} \|\tau\|_{1,p}).$$

Next, define

$$(2.12) \quad \theta_0(x) = \theta(x) - \bar{\theta}.$$

Since $\bar{\theta}_0 = 0$, there exists a unique solution $(u, \sigma) \in W_0^{2,p} \times \bar{W}^{-1,p}$ of the linear Stokes problem

$$(2.13) \quad \begin{cases} -\mu\Delta u + k\nabla\sigma = F + v\nabla\theta_0, \\ \operatorname{div} u = \theta_0, & \text{in } \Omega, \\ u|_{\Gamma} = 0. \end{cases}$$

Moreover (see [1], [4])

$$(2.14) \quad \mu \|u\|_{2,p} + k \|\sigma\|_{1,p} \leq c(|F|_p + (\mu + |v|) \|\theta_0\|_{1,p}) .$$

Since (2.2) holds, and $\|\theta_0\|_{1,p} \leq c \|\theta\|_{1,p}$, one easily gets

$$(2.15) \quad \mu \|u\|_{2,p} + k \|\sigma\|_{1,p} \leq \frac{k}{2} \|\tau\|_{1,p} + c(1 + \frac{\mu+|v|}{\mu+v}) |F|_p + c_2 \frac{\mu+|v|}{m} \|g\|_{1,p} .$$

At this point, we call attention to the sequence of linear maps

$$(F, g, \tau) \rightarrow (F, G) \rightarrow (F, \lambda) \rightarrow (F, \theta) \rightarrow (F, \theta_0) \rightarrow (u, \sigma) ,$$

where F is left unchanged, and the elements $G, \lambda, \theta, \theta_0, (u, \sigma)$, are defined by equations (2.5), (2.4), (2.10), (2.12), and (2.13), respectively. The product map is linear and continuous, by (2.15). Hence, if (u_1, σ_1) is the solution corresponding to the data (F, g, τ_1) , it follows that $(u - u_1, \sigma - \sigma_1)$ is the solution corresponding to the data $(0, 0, \tau - \tau_1)$: Consequently, $\|\sigma - \sigma_1\|_{1,p} \leq (1/2) \|\tau - \tau_1\|_{1,p}$, and the map $\tau \rightarrow \sigma$ is a contraction in $\bar{W}^{1,p}$. Hence, it has a (unique) fixed point $\sigma = \tau$.

Now, we prove that the pair (u, σ) , corresponding to the fixed point $\sigma = \tau$ is a solution of (2.1). The main point is to prove $(2.1)_2$, since $(2.1)_1$ and $(2.1)_3$ follow immediately from (2.13). From $(2.10)_1$ we get

$$(2.16) \quad \lambda = \frac{\mu+v}{k} \Delta \operatorname{div} u + \frac{1}{k} \operatorname{div} F ,$$

since $\Delta \theta = \Delta \theta_0 = \Delta \operatorname{div} u$. On the other hand, by applying the divergence operator to both sides of equation $(2.13)_1$, and by using $(2.13)_2$ and (2.16), we show that $\lambda = \Delta \sigma$. Now, we replace λ by the right hand side of (2.16) in the first term on the left hand side of (2.8), and by $\Delta \sigma$ in the second term on the left hand side of (2.8). This yields

$$(2.17) \quad \begin{aligned} & \langle \varphi, m \Delta \operatorname{div} u \rangle - \langle \operatorname{div}(\varphi v), \Delta \sigma \rangle = \\ & = \langle \varphi, \Delta g - 2 \nabla v : \nabla^2 \sigma - \Delta v \nabla \sigma - \Delta(\sigma \operatorname{div} v) \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega) . \end{aligned}$$

We claim that

$$(2.18) \quad \begin{aligned} & - \langle \operatorname{div}(\varphi v), \Delta \sigma \rangle + \langle \varphi, 2 \nabla v : \nabla^2 \sigma + \Delta v \nabla \sigma \rangle = \\ & = \langle \varphi, \Delta(v \nabla \sigma) \rangle, \quad \forall \varphi \in \mathcal{D}(\Omega) , \end{aligned}$$

where the pairing on the right hand side denotes the duality between $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$.

Let $\sigma_n \in C^3(\Omega)$, $\sigma_n \rightarrow \sigma$ in $W^{1,p}(\Omega)$, as $n \rightarrow \infty$. The identity (2.18), for the functions σ_n , follows from the formulae $\Delta(v \nabla \sigma_n) = v \nabla \Delta \sigma_n + 2 \nabla v : \nabla^2 \sigma_n + \Delta v \nabla \sigma_n$. By passing to the limit as $n \rightarrow \infty$, we prove (2.18) for σ .

From (2.17) and (2.18) we get

$$(2.19) \quad \Delta(m \operatorname{div} u + m\bar{\theta} + v \cdot \nabla \sigma + \sigma \operatorname{div} v - g) = 0, \quad$$

in Ω . Assume that one has

$$(2.20) \quad m \operatorname{div} u + m\bar{\theta} + v \cdot \nabla \sigma + \sigma \operatorname{div} v - g = 0$$

in Ω . Then, by integrating in Ω both sides of this equation, one easily concludes that $\bar{\theta} = 0$. Hence (2.1)₂ holds, as desired. Let us prove (2.20)⁽²⁾. Set

$\Omega_\epsilon = \{z \in \Omega : \operatorname{dist}(z, \Gamma) > \epsilon\}$, $\Gamma_\epsilon = \{z \in \Omega : \operatorname{dist}(z, \Gamma) = \epsilon\}$, where $\epsilon > 0$ is assumed to be enough small so that Γ_ϵ is a regular manifold. Let $g_\epsilon(x, y)$ be the Green's function for the Dirichlet problem in Ω_ϵ , and let $x \in \Omega$ be fixed. We want to prove (2.20) at x .

For $\epsilon \in]0, (1/2)\operatorname{dist}(x, \Gamma)[$ one has $\sup |\nabla g_\epsilon(x, y)| < K$, for $y \in \Gamma_\epsilon$, where K does not depend on ϵ . Let U denote the left hand side of (2.20). Since U is harmonic in Ω_ϵ , one has $U(x) = \int (\partial g_\epsilon(x, y) / \partial n_y) U(y) d\Gamma_\epsilon(y)$. This shows that $|U(x)| < K \int |U(y)| d\Gamma_\epsilon$. Consequently, if we show that $\liminf \int |U(y)| d\Gamma_\epsilon = 0$, as $\epsilon \rightarrow 0$, it must be that $U(x) = 0$. Moreover, it is sufficient to verify that

$$(2.21) \quad \liminf_{\epsilon \rightarrow 0} \operatorname{ess} \int_{\Gamma_\epsilon} |v \cdot \nabla \sigma| d\Gamma_\epsilon = 0, \quad$$

since the function $m \operatorname{div} u + m\bar{\theta} + \sigma \operatorname{div} v - g \in W_0^{1,p}(\Omega)$ converges uniformly to 0 as $\operatorname{dist}(y, \Gamma) \rightarrow 0$.

Assume by contradiction, that there exist positive constants ϵ_0 and δ such that the integral on the left hand side of (2.21) is greater than δ , for a.a. $\epsilon \in]0, \epsilon_0[$. Denoting by K_0 the Lipschitz constant of v in $\bar{\Omega}$, and recalling that v vanishes on Γ , it follows that $\int |\nabla \sigma| d\Gamma_\epsilon > \delta / (K_0 \epsilon)$, for almost all $\epsilon \in]0, \epsilon_0[$. This contradicts the estimate

$$\int_0^{\epsilon_0} \left(\int_{\Gamma_\epsilon} |\nabla \sigma| d\Gamma_\epsilon \right) d\epsilon = \int_{\Omega \setminus \Omega_{\epsilon_0}} |\nabla \sigma| dy < +\infty.$$

⁽²⁾ In case that $j > 0$ (see theorem 1.1) the proof of (2.20) is immediate, since the left hand side of (2.20) is harmonic in Ω and vanishes on Γ (since $\operatorname{div} u = -\delta$, $v = 0$, $\operatorname{div} v = g = 0$, on Γ). However, if $j = -1$, this argument has to be carefully handled, since the function $\nabla \sigma$ has not a trace on Γ .

The existence part of theorem 2.1 is completely proved. As in [2], section 2, we show that there exists a constant c_0 such that the solution of (2.1) is unique if $\|v\|_{2,p} < (\mu_0 k) / [c_0(u+|v|)^2]$, where $\mu_0 = \min\{\mu, \mu + v\}$.

For convenience, we set

$$(2.22) \quad \gamma = \min \left\{ \frac{c_1 m}{\mu + v}, \frac{m}{2c_2(u+|v|)}, \frac{\mu_0}{c_0(u+|v|)^2} \right\}.$$

Then, inequality (2.2) includes all the assumptions on $\|v\|_{2,p}$ utilized in proving theorem 2.1.

3. The non-linear problem

In this section we prove the existence and uniqueness theorem 3.1 in case that $j = -1$, i.e. we prove theorem 3.1 below. We close the Section by showing how to adapt the proofs, in case the $j > 0$.

Recalling (1.4), we set

$$T_i = \sup_{I(l, l_1)} |\omega_i(\sigma, \alpha)|, \quad S_i = \sup_{I(l, l_1)} \frac{|\omega_i(\sigma, \alpha) - \omega_i(\tau, \beta)|}{|(\sigma, \alpha) - (\tau, \beta)|}, \quad i = 1, 2.$$

Moreover, we set $\tilde{\omega}_1(t, s) = \sup |\omega_1(\sigma, \alpha)|$, for $|\sigma| < t$, $|\alpha| < s$. We denote by c_3 a positive constant such that

$$(3.1) \quad |\tau|_\infty < c_3 \|\tau\|_{1,p}, \quad |\beta|_\infty < c_3 \|\beta\|_{1,p}, \quad \forall \tau \in \bar{W}^{1,p}, \quad \forall \beta \in W_0^{1,p}.$$

For convenience, in this section we denote by $c', c_i', i > 0$, positive constants depending at most on $\Omega, n, p, \mu, \nu, k, m, \zeta_0, c_\nu, \chi, \chi_0, \chi_1, l, l_1, T_1, S_1$. The symbol c' may be utilized, even in the same equation, to denote distinct constants.

One has the following result⁽³⁾:

Theorem 3.1. Let $p > n$ be fixed. There exist constants c_0', c_1' , such that if $f \in L^p$, $g \in \bar{W}_0^{1,p}$, $h \in L^p$, and

$$(3.2) \quad \|f\|_p + \|g\|_{1,p} + \|h\|_p < c_0',$$

there exists a unique solution $(u, \sigma, \alpha) \in W_0^{2,p} \times \bar{W}^{1,p} \times W_0^{2,p}$ of problem (1.5), in the ball

$$(3.3) \quad \|u\|_{2,p} + \|\sigma\|_{1,p} + \|\alpha\|_{2,p} < c_1'.$$

Hence, there exists a unique solution $(u, \rho, \zeta) \in W_0^{2,p} \times W^{1,p} \times W^{2,p}$ of problem (1.1), (1.3), in the corresponding neighborhood of $(0, m, \zeta_0)$.

In order to prove theorem 3.1 we state some auxiliary results. The verification of the following statement is immediate, and is left to the reader:

Lemma 3.1. Let $\tau \in \bar{W}^{1,p}$, $\beta \in W_0^{1,p}$, verify the assumptions

⁽³⁾We assume here that $p(\rho, \zeta)$ has Lipschitz continuous first order derivatives. However, theorem 3.1 still holds if $p(\rho, \zeta)$ is only assumed to be continuously differentiable.

$$(3.4) \quad \|\tau\|_{1,p} < \frac{\ell}{c_3}, \quad \|\beta\|_{1,p} < \frac{\ell_1}{c_3},$$

and let $v \in W^{2,p}$, $f \in L^p$, $g \in \bar{W}_0^{1,p}$, $h \in L^p$. Then

$$(3.5) \quad \begin{cases} |F(f,v,\tau,\beta)|_p < \frac{3}{2} m(|f|_p + c\|v\|_{1,p}^2) + \tilde{\omega}_1(|\tau|_\infty, |\beta|_\infty) \|\tau\|_{1,p} + T_2 \|\beta\|_{1,p}, \\ |H(h,v,\tau,\beta)|_p < \frac{3}{2} m|h|_p + c c_v m\|v\|_{1,p} \|\beta\|_{1,p} + \\ + c(\chi_0 + \chi_1) \|v\|_{1,p} \|v\|_{2,p} + c \frac{\zeta_0 T_2}{m} (\|v\|_{2,p} \|\tau\|_{1,p} + |g|_p). \end{cases}$$

Theorem 2.1, lemma 3.1, and classical results for the Dirichlet problem (3.7), yield the following statement:

Theorem 3.2. Let $v \in W_{0,d}^{2,p}$ verify (2.2), and assume that the hypotheses in lemma 3.1 hold. Then, there exists a unique solution $(u, \sigma) \in W_{0,d}^{2,p} \times \bar{W}^{1,p}$ of problem

$$(3.6) \quad \begin{cases} -\mu \Delta u - v \nabla \operatorname{div} u + k \nabla \sigma = F(f,v,\tau,\beta), \\ m \operatorname{div} u + v \cdot \nabla \sigma + \sigma \operatorname{div} v = g, \text{ in } \Omega, \\ u|_\Gamma = 0. \end{cases}$$

and a unique solution $\alpha \in W_0^{2,p}$ of the problem

$$(3.7) \quad \begin{cases} -\chi \Delta \alpha = H(h,v,\tau,\beta), \text{ in } \Omega, \\ \alpha|_\Gamma = 0. \end{cases}$$

Moreover,

$$(3.8) \quad \begin{aligned} \mu \|u\|_{2,p} + k \|\sigma\|_{1,p} &< c(1 + \frac{\mu + |v|}{\mu + \nu}) [m|f|_p + \|v\|_{1,p}^2 + \\ &+ \tilde{\omega}_1(|\tau|_\infty, |\beta|_\infty) \|\tau\|_{1,p} + T_2 \|\beta\|_{1,p}] + c \frac{\mu + |v|}{m} |g|_{1,p}, \end{aligned}$$

$$(3.9) \quad \begin{aligned} \chi \|\alpha\|_{2,p} &< c|h|_p + c c_v m\|v\|_{1,p} \|\beta\|_{1,p} + c(\chi_0 + \chi_1) \|v\|_{2,p}^2 + \\ &+ c \frac{\zeta_0 T_2}{m} (\|v\|_{2,p} \|\tau\|_{1,p} + |g|_p). \end{aligned}$$

In the sequel, the solution (u, σ, α) of problems (3.6), (3.7) with data (v, τ, β) is denoted by $(u, \sigma, \alpha) = T(v, \tau, \beta)$. Let us write (3.8), (3.9) in the abbreviated form

$$(3.10) \quad \begin{cases} \|u\|_{2,p} + \|\sigma\|_{1,p} < c'_2(\|f\|_p + \|g\|_{1,p} + \|v\|_{2,p}^2 + \\ \quad + \|\tau\|_{1,p}^2 + \|\beta\|_{1,p}^2) + c'_3\|\beta\|_{1,p} , \\ \|a\|_{2,p} < c'_2(\|h\|_p + \|g\|_{1,p} + \|v\|_{2,p}^2 + \|\tau\|_{1,p}^2 + \|\beta\|_{1,p}^2) . \end{cases}$$

Set

$$(3.11) \quad r_0 = \min\left\{\gamma k, \frac{\ell}{c_3}, \left(\frac{\ell_1}{5 c'_2 c_3}\right)^{1/2}, \frac{1}{25 c'_2}, \frac{1}{5 c'_2(1+c'_3)}\right\} ,$$

let $r \in]0, r_0]$, and assume that

$$(3.12) \quad \|f\|_p < r^2, \|h\|_p < r^2, \|g\|_{1,p} < r^2 ,$$

and that

$$(3.13) \quad \|v\|_{2,p} < r, \|\tau\|_{1,p} < r, \|\beta\|_{1,p} < 5 c'_2 r^2 .$$

Under these assumptions, the conditions (2.2), (3.4) are fulfilled. Consequently, from (3.10) it follows that $(u, \sigma, \alpha) = T(v, \tau, \beta)$ verifies

$$(3.14) \quad \|u\|_{2,p} < r, \|\sigma\|_{1,p} < r, \|a\|_{1,p} < 5 c'_2 r^2 .$$

Consequently $T(B_r) \subset B_r$, where B_r is the subset of $W_{0,d}^{2,p} \times \overline{W}^{1,p} \times W_0^{2,p}$ defined by equations (3.13). In order to accomplish the proof of theorem 3.1, we will show that T is a contraction in B_r , respect to the norm $W_0^{1,2} \times L^2 \times W_0^{1,2}$, if r is sufficiently small. It follows then, by the contracting mapping principle, that T has a (unique) fixed point $(u, \sigma, \alpha) = (v, \tau, \beta)$ in B_r (we could also use the Schauder's fixed point theorem, since B_r is a convex and compact subset in respect to the $W_0^{1,2} \times L^2 \times W_0^{1,2}$ topology).

Let $(u, \sigma, \alpha) = T(v, \tau, \beta)$, $(u_1, \sigma_1, \alpha_1) = T(v_1, \tau_1, \beta_1)$, $F = F(f, v, \tau, \beta)$, $F_1 = F(f, v_1, \tau_1, \beta_1)$, $H = H(h, v, \tau, \beta)$, $H_1 = H(h, v_1, \tau_1, \beta_1)$. One has, in Ω ,

$$(3.15) \quad \begin{cases} -\mu \Delta(u-u_1) - v \nabla \operatorname{div}(u-u_1) + k \nabla(\sigma-\sigma_1) = F - F_1, \\ m \operatorname{div}(u-u_1) + v_1 \cdot \nabla(\sigma-\sigma_1) + (v-v_1) \cdot \nabla \sigma + \\ + \sigma_1 \operatorname{div}(v-v_1) + (\sigma-\sigma_1) \operatorname{div} v = 0, \end{cases}$$

and

$$(3.16) \quad -\chi \Delta(u-u_1) = H - H_1.$$

Here, we use the notation $\| \cdot \|_{k,2} = \| \cdot \|_k$. By multiplying both sides of equation (3.15)₁ by $m(u-u_1)$, both sides of equation (3.15)₂ by $k(\sigma-\sigma_1)$, by integrating the resulting equation in Ω , and by adding them side by side, we show that

$$(3.17) \quad \begin{aligned} m(u-v) \|\nabla(u-u_1)\|_0^2 &< m \|F-F_1\|_{-1} \|u-u_1\|_1 + \\ &+ ck \|\sigma_1\|_{1,p} \|v-v_1\|_1 \|\sigma-\sigma_1\|_0 + ck \|v\|_{2,p} \|\sigma-\sigma_1\|_0^2. \end{aligned}$$

In proving (3.17) we utilize some well known Sobolev inequalities. Note that if $2^* = 2n/(n-2)$, one has $(1/2^*) + (1/2) + (1/p) < 1$.

Arguing as on proving equation (3.12) in reference [2], we show that

$$(3.18) \quad \begin{aligned} \|u-u_1\|_1^2 + c'_4(1 - c'_5 \|v\|_{2,p}) \|\sigma-\sigma_1\|_0^2 &< \\ &< c'_1 \|\sigma_1\|_{1,p}^2 \|v-v_1\|_1^2 + c'_1 \|F-F_1\|_{-1}^2. \end{aligned}$$

Moreover,

$$(3.19) \quad \|u-u_1\|_1 < c' \|H-H_1\|_{-1}.$$

Straightforward calculations show that (see appendix)

$$(3.20) \quad \begin{aligned} \|F-F_1\|_{-1} &< c'(\|f\|_p + \|v_1\|_{1,p}^2) \|\tau-\tau_1\|_0 + \\ &+ c'(1 + \|\tau\|_{1,p})(\|v\|_{1,p} + \|v_1\|_{1,p}) \|v-v_1\|_1 + \\ &+ c'(\|\tau\|_{1,p} + \|\tau_1\|_{1,p} + \|\beta\|_{1,p} + \|\beta_1\|_{1,p})(\|\tau-\tau_1\|_0 + \|\beta-\beta_1\|_0), \end{aligned}$$

and that

$$\begin{aligned}
\|H-H_1\|_{-1} &\leq c'(\|v\|_{1,p} + \|v_1\|_{1,p} + \|\tau\|_{1,p} + \|B\|_{1,p}) \|v-v_1\|_1 + \\
&+ c'(|h|_p + |g|_p + (\|B\|_{1,p} + \|\tau\|_{1,p})\|v\|_{1,p} + (1 + \|\tau_1\|_{1,p} + \|B_1\|_{1,p} + \\
(3.21) \quad &+ \|\tau_1\|_{1,p}^3 + \|B_1\|_{1,p}^3)\|v_1\|_{1,p})\|\tau-\tau_1\|_0 + c'(|g|_p + \|v\|_{1,p}\|\tau\|_{1,p} + \\
&+ \|v_1\|_{1,p}\|\tau_1\|_{1,p} + \|v_1\|_{1,p})\|B-B_1\|_0 \dots
\end{aligned}$$

By using (3.12), (3.13) it follows that the coefficients of $\|v-v_1\|_1$, $\|\tau-\tau_1\|_0$, $\|B-B_1\|_0$, in the right hand sides of equations (3.20), (3.21), are polynomials on r , vanishing for $r = 0$, and with coefficients which are positive constants of type c' . Similarly, the coefficients $c'_5\|v\|_{2,p}$ and $c'\|\sigma_1\|_{1,p}^2$ (that appear in equation (3.18)) are polynomials on r of the above type. Since the exact form of these polynomials is not important here, we will denote them by the symbol $\varepsilon = \varepsilon(r)$.

By using the above notation, the estimates (3.18), (3.19), (3.20), (3.21), yield

$$(3.22) \quad \|u-u_1\|_1^2 + (c'-\varepsilon)\|\sigma-\sigma_1\|_0^2 \leq \varepsilon\|v-v_1\|_1^2 + \varepsilon\|\tau-\tau_1\|_0^2 + \varepsilon\|B-B_1\|_0^2,$$

$$(3.23) \quad \|\alpha-\alpha_1\|_1^2 \leq \varepsilon\|v-v_1\|_1^2 + \varepsilon\|\tau-\tau_1\|_0^2 + \varepsilon\|B-B_1\|_0^2.$$

Hence,

$$\begin{aligned}
(3.24) \quad &\|u-u_1\|_1^2 + (c'-\varepsilon)\|\sigma-\sigma_1\|_0^2 + \|\alpha-\alpha_1\|_1^2 \leq \\
&\leq \varepsilon\|v-v_1\|_1^2 + \varepsilon\|\tau-\tau_1\|_0^2 + \varepsilon\|B-B_1\|_0^2.
\end{aligned}$$

Hence, for a sufficiently small value of r (depending only on $\Omega, n, p, \dots, T_1, S_1$), T is a contraction in B_T , with respect to a suitable norm in $W_0^{1,2} \times L^2 \times W_0^{1,2}$. The proof of theorem 3.1 is accomplished. □

Proof of theorem 1.1. The changes to be made on the proof of theorem 3.1, in order to adapt it for all values of j , are quite obvious. We start by showing it for theorem 2.1, which is the main tool in our proofs.

The proof of theorem 2.1 was done by solving the sequence of linear problems (2.5), (2.4), (2.10), (2.12), (2.13), and by getting suitable estimates for the corresponding

solutions. Let us show that the proof goes on, in case that $j > 0$, exactly as for $j = -1$.

Problem (2.5): The following estimate for the solution of problem (2.5) is easily obtained by using Sobolev's inequalities:

$$\|G\|_{j,p} \leq \|g\|_{j+2,p} + \frac{m}{\mu+\nu} \|F\|_{j+1,p} + c\|v\|_{j+3,p} \|\tau\|_{j+2,p}.$$

Problem (2.4): The following result is proved in [3]: There exist positive constants c and c_1 (here the constants depend also on j) such that if $\|v\|_{j+3,p} \leq c_1 mk/(\mu+\nu)$, and $v = 0$ on Γ , then there exists a solution λ of problem (2.4), moreover one has

$$\frac{mk}{2(\mu+\nu)} \|\lambda\|_{j,p} \leq c\|G\|_{j,p}.$$

Problem (2.10): For the solution θ of the Dirichlet problem (2.10) the estimate

$$(\mu+\nu)\|\theta\|_{j+2,p} \leq c(\|F\|_{j+1,p} + k\|\lambda\|_{j,p})$$

is well known.

Problem (2.12): Obviously, the function θ_0 , defined by equation (2.12), verifies the estimate $\|\theta_0\|_{j+2,p} \leq c\|\theta\|_{j+2,p}$.

Problem (2.13): The solution (u, σ) of this Stokes linear problem (see [1], [4]) verifies the estimate

$$\mu\|u\|_{j+3,p} + k\|\sigma\|_{j+2,p} \leq c(\|F\|_{j+1,p} + (\mu+|v|)\|\theta_0\|_{j+2,p}).$$

By putting together the above estimates, one easily gets (c.f.r. (2.15))

$$\begin{aligned} \mu\|u\|_{j+3,p} + k\|\sigma\|_{j+2,p} &\leq \frac{k}{2} \|\tau\|_{j+2,p} + \\ &+ c(1 + \frac{\mu+|v|}{\mu+\nu}) \|F\|_{j+1,p} + c_2 \frac{\mu+|v|}{m} \|g\|_{j+2,p}. \end{aligned}$$

By arguing as in section 2 (recall the footnote (1)) one proves the following result:

Proposition 3.3. Let p, j , and Γ be as in theorem 1.1 and let $F \in W^{j+1,p}$, $g \in \bar{W}_0^{j+2,p}$. Then, there exist positive constants $c = c(\Omega, n, p, j)$, $\gamma = \gamma(\Omega, n, p, j, \mu, \nu, m)$, such that if $v \in W_{0,d}^{j+3,p}$ verifies the condition $\|v\|_{j+3,p} \leq \gamma k$, then there exists a unique solution $(u, \sigma) \in W_{0,d}^{j+3,p} \times \bar{W}^{j+2,p}$ of problem (2.1). Moreover,

$$\mu\|u\|_{j+3,p} + k\|\sigma\|_{j+2,p} \leq c(1 + \frac{\mu+|v|}{\mu+\nu}) \|F\|_{j+1,p} + c \frac{\mu+|v|}{m} \|g\|_{j+2,p}.$$

The rest of the proof of Theorem 1.1 (the "non-linear part") closely follows the proof of Theorem 3.1. For an arbitrary $j > -1$, we define the set B_r by the inequalities (c.f.r. (3.13))

$$|v|_{j+3,p} \leq r, \quad |u|_{j+2,p} \leq r, \quad |B|_{j+2,p} \leq c'r^2,$$

and we prove that (for sufficiently small values of r) $T(B_r) \subset B_r$, and that T is a contraction in B_r with respect to the norm $W_0^{1,2} \times L^2 \times W_0^{1,2}$. The proofs of these two statements in case that $j > -1$, differ from the proofs given above only by the particular Sobolev's inequalities to be used.

Before ending this section we remark that in Theorem 1.1, if $h \in W^{j,p}$ (instead of $h \in W^{j+1,p}$) then $\zeta \in W^{j+2,p}$, and if $h \in W^{j+2,p}$ then $\zeta \in W^{j+4,p}$.

4. The incompressible limit.

In this section we study the system

$$\begin{aligned} -\mu \Delta u - \nu \nabla \operatorname{div} u + \nabla p(\rho) &= \rho [f - (u \cdot \nabla) u] , \\ \operatorname{div}(\rho u) &= 0, \text{ in } \Omega , \\ u|_{\Gamma} &= 0 , \end{aligned} \quad (4.1)$$

describing the barotropic motion of a compressible, viscous fluid. Since we are interested on studying the limit of the solution u when $k \equiv p'_\rho(m) \rightarrow +\infty$, it is necessary to state an existence result for problem (4.1) in which: (i) the dependence of some suitable structural constants of the state function $p(\rho)$ in terms of k is given; (ii) the dependence on k of the constants appearing on the estimates, is shown. This is the aim of theorem 4.2 below.

Let $p(\rho)$ be continuously differentiable in $[m - (l/k), m + (l/k)]$, where $0 < l/k < m/2$, and write p'_ρ in the form $p'_\rho(m+\sigma) = k + \omega(\sigma)$. We assume that there exists $\alpha \in]0, 1]$ such that

$$|\omega(\sigma)| \leq S|\sigma|^\alpha, \quad \forall \sigma \in [-l/k, l/k] , \quad (4.2)$$

and we denote by ϕ a positive constant for which

$$S \leq \phi k^{1+\alpha} . \quad (4.3)$$

Arguing as in proving theorem 3.2, we show

Theorem 4.1. Let the above assumptions on the state function $p(\rho)$ hold, let $v \in W^{2,p}_{0,d}$ verify (2.2), and let

$$\|k\tau\|_{1,p} < \frac{l}{c_3} . \quad (4.4)$$

If $f \in L^p$, there exists a unique solution $(u, \sigma) \in W^{2,p}_{0,d} \times \bar{W}^{1,p}$ of the problem

$$(4.5) \quad \begin{cases} -\mu \Delta u - \nu \nabla \operatorname{div} u + k \nabla \sigma = F(f, v, \tau) \\ m \operatorname{div} u + v \cdot \nabla \sigma + \sigma \operatorname{div} v = 0, \text{ in } \Omega , \\ u|_{\Gamma} = 0 , \end{cases}$$

where $F(f, v, \tau) = (\tau+m)[f - (v \cdot \nabla)v] - \omega(\tau)\nabla \tau$. Moreover,

$$\|u\|_{2,p} + \|k\sigma\|_{1,p} \leq c_1 \|f\|_p + c_2 \|v\|_{1,p}^2 + c_3 \phi \|k\tau\|_{1,p}^{1+\alpha} . \quad (4.6)$$

Here the positive constants c_1^i, c_2^i, c_3^i , depend at most on $\Omega, n, p, \mu, \nu, m$ (and are distinct from the constants c_1^i, c_2^i, c_3^i , in section 3). Following section 3, we denote by $(u, \sigma) = T(v, \tau)$ the solution of system (4.5). Define

$$(4.7) \quad r_0 = \min\left\{\gamma k, \frac{\ell}{c_3}, \frac{1}{3c_2^i}, \left(\frac{1}{3c_3^i}\right)^{1/\alpha}\right\},$$

and assume that $f \in L^p$ is fixed, and verifies

$$(4.8) \quad \|f\|_p < r/(3c_1^i).$$

One easily shows that $T(B_r) \subset B_r$, where B_r is defined by the inequalities

$$(4.9) \quad \|v\|_{2,p} < r, \|k\tau\|_{1,p} < r.$$

Moreover, there exist positive constants c^i, c_4^i, c_5^i , depending at most on $\Omega, n, p, \mu, \nu, m$, such that

$$(4.10) \quad \begin{aligned} \|u - u_1\|_1^2 + c_4^i(1 - c_5^i\|v\|_{2,p})\|k(\sigma - \sigma_1)\|_0^2 &< \\ &< c^i\|w_1\|_{1,p}^2\|v - v_1\|_1^2 + c^i\|F - F_1\|_{-1}^2, \end{aligned}$$

where $(u_1, \sigma_1) = T(v_1, \tau_1)$, $F = F(f, v, \tau)$, $F_1 = F(f, v_1, \tau_1)$. Inequality (4.10) is proved as inequality (3.18). Straightforward calculations (see (3.20)) show that

$$(4.11) \quad \begin{aligned} \|F - F_1\|_{-1} &\leq c\|f\|_p\|\tau - \tau_1\|_0 + c(1 + \|\tau\|_{1,p})(\|v\|_{1,p} + \|v_1\|_{1,p})\|v - v_1\|_1 + \\ &\quad + c\|v_1\|_{1,p}^2\|\tau - \tau_1\|_0 + \\ &\quad + c_3^\alpha(\|k\tau\|_{1,p}^\alpha + \|k\tau_1\|_{1,p}^\alpha)\|k(\tau - \tau_1)\|_0. \end{aligned}$$

Arguing as in section 3, one easily verifies that there exists a positive constant $r_1 < r_0$, depending only on $\Omega, n, p, \mu, \nu, m, \ell, \phi, k_0$, such that for $r < r_1$, T is a contraction in B_r , with respect to the $W_0^{1,2} \times L^2$ norm (for convenience, we denote by k_0 a positive constant such that $k > k_0$). The above result proves the following theorem:

Theorem 4.2. Let $p > n$ be fixed, and let the above assumptions on the state function $p(\rho)$ hold. Then, there exist positive constants c_6^i, c_7^i , depending only on

$\Omega, n, p, \nu, \nu, m, \ell, \phi$ and k_0 , such that if $f \in L^p$ verifies $|f|_p < c_6^1$, then there exists a unique solution $(u, \rho) \in W_0^{2,p} \times W^{1,p}$ of problem (4.1), (1.3)₁, in the ball $|u|_{2,p} < c_7^1, |\rho - m|_{1,p} < c_7^1/k$.

From now on we assume that $p_\lambda(\rho)$ is a family of state functions depending on a parameter $\lambda \in [\lambda_0, +\infty[$, and such that $k_\lambda \rightarrow +\infty$ as $\lambda \rightarrow +\infty$. Our assumptions on $p_\lambda(\rho)$ are the following:

Let $Dp_\lambda(\rho)$ denote the derivative $dp_\lambda(\rho)/d\rho$. We set $k_\lambda = Dp_\lambda(m)$, we assume that $k_\lambda > k_0 > 0$, and we suppose that $p_\lambda(\rho)$ is defined and continuously differentiable in the interval $[m - \ell/k_\lambda, m + \ell/k_\lambda]$, where $0 < \ell < (k_0 m)/2$. Moreover, we assume that $|Dp_\lambda(\rho) - Dp_\lambda(m)| < S_\lambda |\rho - m|^\alpha$, for a fixed $\alpha \in]0, 1[$. Hence, by setting $Dp_\lambda(m + \sigma) = k_\lambda + \omega_\lambda(\sigma)$, one has $\omega_\lambda(0) = 0$, and

$$(4.12) \quad |\omega_\lambda(\sigma)| < S_\lambda |\sigma|^\alpha, \quad \forall \sigma \in [-\ell/k_\lambda, \ell/k_\lambda] .$$

We assume that there exists a positive constant ϕ such that

$$(4.13) \quad S_\lambda < \phi k_\lambda^{1+\alpha}, \quad \forall \lambda > \lambda_0 .$$

These quite general assumptions contain the physically interesting cases described by Klainerman and Majda in [5]. In particular, if k, α and S are the parameters relative to a given state function $p(\rho)$, and if we define $p_\lambda(\rho) = \lambda^2 p(\rho)$, then $S_\lambda < (S/k)k_\lambda$. Hence, $S_\lambda < \phi k_\lambda^{1+\alpha}$, where $\phi = S/(\lambda_0^{2\alpha} k^{1+\alpha})$. In that example we may view $M = 1/\lambda$ as the Mach number (see [5]). One has the following result:

Theorem 4.3. Let $p > n$ be fixed, and let the above assumptions on the family of state functions $p_\lambda(\rho)$ hold. Then, there exist positive constants c_8^1, c_9^1 , depending at most on $\Omega, n, p, \nu, \nu, m, \ell, \phi, k_0$, such that if $f \in L^p, |f|_p < c_8^1$, then the following statements hold:

(i) for each $\lambda > \lambda_0$, the problem

$$(4.14) \quad \begin{cases} -\mu \Delta u_\lambda - \nu \nabla \operatorname{div} u_\lambda + \nabla p_\lambda(\rho_\lambda) = \rho_\lambda [f - (u_\lambda \cdot \nabla) u_\lambda] , \\ \operatorname{div}(\rho_\lambda u_\lambda) = 0, \quad \text{in } \Omega , \\ (u_\lambda)|_\Gamma = 0, \quad \bar{\rho}_\lambda = m , \end{cases}$$

has a unique solution $(u_\lambda, \rho_\lambda) \in W_0^{2,p} \times W^{1,p}$ in the ball $\|u_\lambda\|_{2,p} < c_9^1$,

$$\|\rho_\lambda - m\|_{1,p} < c_9^1/k_\lambda.$$

(ii) If $\lim_{\lambda \rightarrow \infty} k_\lambda = +\infty$ as $\lambda \rightarrow \infty$, then $u_\lambda \rightharpoonup u_\infty$ weakly in $W_0^{2,p}$, $\operatorname{div} u_\lambda \rightarrow 0$ weakly in $W_0^{1,p}$, $\rho_\lambda \rightarrow m$ strongly in $W^{1,p}$, $\nabla p_\lambda(\rho_\lambda) \rightarrow \nabla \pi$ weakly in L^p , where $(u_\infty, \nabla \pi)$ is the unique solution of the incompressible Navier-Stokes equations

$$(4.15) \quad \begin{cases} -\mu \Delta u_\infty + \nabla \pi(x) = m[f - (u_\infty \cdot \nabla) u_\infty] , \\ \operatorname{div} u_\infty = 0 , \text{ in } \Omega , \\ (u_\infty)|_\Gamma = 0 . \end{cases}$$

Proof. Part (i) follows from theorem 4.2. Moreover, the estimates show that there exists $u_\infty \in W_0^{2,p}$ such that $u_\lambda \rightharpoonup u_\infty$, and $\rho_\lambda \rightarrow m$, in the topologies indicated in the statement. The convergence of all the sequence u_λ to u_∞ follows from the uniqueness of the solution of (4.15), which holds if c_9^1 is sufficiently small.

From equation (4.14)₂, one easily verifies that $\operatorname{div} u_\lambda \rightarrow 0$ in L^p . Since $\|\operatorname{div} u_\lambda\|_{1,p} < c_9^1$, one gets the weak convergence in $W_0^{1,p}$. Clearly, $\operatorname{div} u_\infty = 0$.

Finally, we pass to the limit in equation (4.14)₁. One has $\nabla p_\lambda(\rho_\lambda) \rightarrow \mu \Delta u_\infty + m[f - (u_\infty \cdot \nabla) u_\infty]$, weakly in L^p , since: $\Delta u_\lambda \rightarrow \Delta u_\infty$ and $-\nu \nabla \operatorname{div} u_\lambda \rightarrow 0$ weakly in L^p ; $\rho_\lambda \rightarrow m$ strongly in $W^{1,p}$; and $\rho_\lambda (u_\lambda \cdot \nabla) u_\lambda \rightarrow m(u_\infty \cdot \nabla) u_\infty$, strongly in L^p . Obviously the limit of the sequence $\nabla p_\lambda(\rho_\lambda)$ must be of the form $\nabla \pi(x)$, for $\pi \in \bar{W}^{1,p}$.

□

5. The equilibrium solutions

The results described in this section are obtained by using very elementary methods. The proofs are independent of the results of the preceding sections.

By an equilibrium solutions (e.s.) of system (4.1) [resp. (1.1)] we mean a solution (u, ρ) [resp. (u, ρ, ζ)] such that $u = 0$ in Ω , and

$$(5.1) \quad 0 < \operatorname{ess\,inf}_{x \in \Omega} \rho(x) \quad , \quad \operatorname{ess\,sup}_{x \in \Omega} \rho(x) < +\infty .$$

If $h = q = 0$ the two problems are equivalent (set $p(\rho) = p(\rho, \zeta_0)$ in system (4.1)). In the sequel we will consider the system (4.1). We call equilibrium solutions the functions $\rho(x)$, in the class (5.1), such that $(0, \rho(x))$ solves (4.1). It is easily shown that a necessary condition for the existence of the equilibrium solution is that $f = \nabla F$, for some potential F . Moreover, the equilibrium solutions are the solutions of the equation

$$(5.2) \quad \nabla p(\rho(x)) = \rho(x) \nabla F(x), \quad x \in \Omega .$$

Here, we are interested in the study of the e.s. under the effect of arbitrarily large external forces $f = \nabla F$. We will present useful necessary and sufficient conditions for the existence of the equilibrium solution for an arbitrary $F \in L^\infty(\Omega)$.

In the sequel, p is a continuously differentiable real function defined on $\mathbb{R}^+ = \{s \in \mathbb{R} : s > 0\}$, such that $p'(s) > 0$, $\forall s \in \mathbb{R}^+$. We look for equilibrium solutions verifying (5.1) and (1.3)₁, for a fixed $m > 0$. We define

$$(5.3) \quad \pi(s) = \int_m^s t^{-1} p'(t) dt, \quad \forall s \in \mathbb{R}^+ .$$

We denote by $]a, b[$ the range of π , $]a, b[= \pi(\mathbb{R}^+)$. One has $-\infty < a < 0 < b < +\infty$, since $\pi(m) = 0$. We define $\phi = \pi^{-1}$. Clearly, $\phi(]a, b[) = \mathbb{R}^+$. We set $\phi(a) = 0$, $\phi(b) = +\infty$.

Let ρ and F be defined and measurable in Ω , and let ρ verify the assumptions (5.1) and (1.3)₁. Then ρ is said to be a weak-equilibrium solution if there exists a real constant c such that

$$(5.4) \quad \pi(\rho(x)) = F(x) + c, \quad \text{a.e. in } \Omega .$$

Moreover, ρ is said to be a strong-equilibrium solution if $\rho \in W^{1,1}$ and if (5.2)

holds a.e. in Ω . A weak-equilibrium solution is a strong e.s. if and only if $\rho \in W^{1,1}$. Furthermore, $\rho \in W^{1,1}$ if and only if $F \in W^{1,1}$. Hence it is sufficient to take in account the weak formulation (5.4). Note that equation (5.4) shows that ρ and F have the same regularity, if ρ is sufficiently smooth. For the sake of convenience we will concentrate here on the L^∞ case.

Definition 5.1. Let $F \in L^\infty$. A function ρ is called an equilibrium solution of system (4.1) if $\rho \in L^\infty$ and if (5.4), (5.1) and (1.3)₁ hold.

We set $m_0 = \text{ess inf } F$ in Ω , $M_0 = \text{ess sup } F$ in Ω . One has the following result:

Theorem 5.2. Let $F \in L^\infty$, be given. There exists an equilibrium solution $\rho(x)$ if and only if there exists a constant

$$(5.5) \quad c \in]a - m_0, b - M_0[,$$

such that

$$(5.6) \quad \frac{1}{|\Omega|} \int_{\Omega} \Phi(c + F(x)) dx = m .$$

If such a constant exists then the (unique) equilibrium solution is given by

$$(5.7) \quad \rho(x) = \Phi(c + F(x)), \quad \forall x \in \Omega .$$

Proof. The condition (5.6) corresponds to condition (1.3)₁. Uniqueness is shown by noting that the left hand side of (5.6) is a continuous and increasing function of c in the interval (5.5). The details are left to the reader.

□

Theorem 5.3. Under the assumptions of theorem 5.2, there exists an equilibrium solution $\rho(x)$ if and only if

$$(5.8) \quad a - m_0 < b - M_0 ,$$

and

$$(5.9) \quad \frac{1}{|\Omega|} \int_{\Omega} \Phi(a - m_0 + F(x)) dx < m < \frac{1}{|\Omega|} \int_{\Omega} \Phi(b - M_0 + F(x)) dx .$$

In this case the e.s. $\rho(x)$ is given by (5.7), where c is the (unique) solution of (5.5), (5.6).

Proof. Condition (5.7) follows from (5.5). Moreover, (5.6) holds for some $c \in]a-m_0, b-M_0[$ if and only if

$$(5.10) \quad \lim_{c \rightarrow (a-m_0)^+} \frac{1}{|\Omega|} \int_{\Omega} \Phi(c + F(x)) dx < m ,$$

$$(5.11) \quad \lim_{c \rightarrow (b-M_0)^-} \frac{1}{|\Omega|} \int_{\Omega} \Phi(c + F(x)) dx > m .$$

These two conditions are equivalent to (5.9), by a well known generalization of Beppo Levi's theorem to functions taking values in $[0, +\infty]$. Note in particular that the integral on the right hand side of (5.9) is well defined, since $0 < \Phi(b-M_0 + F(x)) < +\infty$, a.e. in Ω .

□

Corollary 5.4. If $a = -\infty$ [resp. $b = +\infty$] then the conditions (5.8) and (5.9)₁ [resp. (5.9)₂] hold, for every $F \in L^{\infty}$. In particular, if $]a, b[=]-\infty, +\infty[$ the equilibrium solution exists for every $F \in L^{\infty}$.

Corollary 5.5. Let $F \in L^{\infty}$. Then, the equilibrium solution exists if

$$(5.12) \quad M_0 - m_0 < \min\{-a, b\} .$$

In particular this holds if $\|F\|_{\infty} < (1/2) \min\{-a, b\}$.

Proof. If (5.12) holds then $a - m_0 + F(x) < 0$, and $b - M_0 + F(x) > 0$, a.e. in Ω . Hence $\Phi(a-m_0 + F(x)) < m < \Phi(b-M_0 + F(x))$, and assumption (5.9) holds.

□

Remark 5.6. The lack of condition (5.9)₁ [resp. (5.9)₂] corresponds to the formation of vacuum [resp. points of infinite density]. If $a = -\infty$ [resp. $b = +\infty$] the first [resp. second] phenomena does not occur, under the effect of bounded potentials F .

Let us now consider the one dimensional case $\Omega =]0, \ell[, \ell > 0$, in the presence of external forces $f \in L^1$. The potential

$$F(x) = \int_0^x f(t) dt$$

is then an absolutely continuous function in $[0, \ell]$. Consequently, the equilibrium solution ρ (if exists) belongs to $W^{1,1}$ and verifies (5.2). The sufficient condition

(5.12) can be written in the equivalent form

$$(5.13) \quad \left| \int_x^y f(t) dt \right| < \min\{-a, b\}, \quad \forall x, y \in [0, l].$$

In particular, $|f|_1 < \min\{-a, b\}$ is also a sufficient condition for the existence of the e.s.

Examples. As shown before, the existence of the equilibrium solution under a given potential F , strongly depends on the values of a and b . It is of interest to consider the classical case $p(\rho) = R\rho^\gamma$, $R > 0$, $\gamma > 0$. It follows immediately from our results that if $\gamma > 1$ [resp. $\gamma < 1$], vacuum [resp. infinite densities] may occur. On the contrary, if $\gamma = 1$ the e.s. exists, for every bounded potential F .

Let us give an example, corresponding to the main case $\gamma > 1$. For convenience, we consider the one dimensional case $\Omega =]0, 1[$. Let $m = 1$, $\gamma = 2$, $R = 1/2$. Then $p(\rho) = (1/2)\rho^2$, $\pi(\rho) = \rho - 1$, $\phi(s) = 1 + s$, $a = -1$, $b = +\infty$. In the light of corollary 5.4, we have only to check the first condition (5.9), which corresponds to the formation of vacuum. By assuming that $f(x) = \beta$ is constant, this condition is $|\beta| < 2$. Formulae (5.6) and (5.7) yield $c = -\beta/2$ and $\rho(x) = 1 + \beta(x - 1/2)$, respectively. If $\beta > 0$, the forces act on the positive direction, and the point $x = 0$ is subject to the largest decompression. For $\beta = 2$ the vacuum is attained at the point $x = 0$, since $\rho(0) = 0$.

Appendix. For the reader's convenience, we will prove here the estimate (3.21). The proof of (3.20) is easier, and is left to the reader. Recall that under the assumptions made in section 3 one has $|\tau|_\infty \leq L$, $|\tau_1|_\infty \leq L$, $|\beta|_\infty \leq L_1$, $|\beta_1|_\infty \leq L_1$.

In the sequel we use freely the inequality

$$(1) \quad \|\psi_1 \psi_2 \psi_3\|_{-1} \leq c \|\psi_1\|_\infty \|\psi_2\|_0 \|\psi_3\|_p,$$

which is easily proved by using the Sobolev's inequality $\|\varphi\|_{2+} \leq c \|\varphi\|_1$, $\forall \varphi \in W_0^{1,2}$. One has

$$(2) \quad \begin{aligned} \|H(h, v, \tau, \beta) - H(h, v_1, \tau_1, \beta_1)\|_{-1} &\leq \|(\tau - \tau_1)h\|_{-1} + c_v \|(\tau - \tau_1)v \cdot \nabla \beta\|_{-1} + \\ &+ \|(m + \tau_1)(v - v_1) \cdot \nabla \beta\|_{-1} + \|(m + \tau_1)v_1 \cdot \nabla (\beta - \beta_1)\|_{-1} + \\ &+ \|\psi(v, v) - \psi(v_1, v_1)\|_{-1} + \|(\beta - \beta_1)(m + \tau)^{-1} \omega_2(\tau, \beta)(v \cdot \nabla \tau - g)\|_{-1} + \\ &+ \|(\zeta_0 + \beta_1)(\tau_1 - \tau)(m + \tau)^{-1} (m + \tau_1)^{-1} \omega_2(\tau, \beta)(v \cdot \nabla \tau - g)\|_{-1} + \\ &+ \|(\zeta_0 + \beta_1)(m + \tau_1)^{-1} (\omega_2(\tau, \beta) - \omega_2(\tau_1, \beta_1))(v \cdot \nabla \tau - g)\|_{-1} + \\ &+ \|(\zeta_0 + \beta_1)(m + \tau_1)^{-1} \omega_2(\tau_1, \beta_1)(v - v_1) \cdot \nabla \tau\|_{-1} + \\ &+ \|(\zeta_0 + \beta_1)(m + \tau_1)^{-1} \omega_2(\tau_1, \beta_1)v_1 \cdot \nabla (\tau - \tau_1)\|_{-1}. \end{aligned}$$

By using adequately the inequality (1), one gets

$$(3) \quad \begin{aligned} \|H - H_1\|_{-1} &\leq c \|h\|_p \|\tau - \tau_1\|_0 + c c_v \|v\|_{1,p} \|\beta\|_{1,p} \|\tau - \tau_1\|_0 + \\ &+ c(m + L) \|\beta\|_{1,p} \|v - v_1\|_0 + \|(m + \tau_1)v_1 \cdot \nabla (\beta - \beta_1)\|_{-1} + \\ &+ c(\chi_0 + \chi_1)(\|v\|_{1,p} + \|v_1\|_{1,p}) \|v - v_1\|_1 + \\ &+ c(2/m) T_2(\|v\|_{1,p} \|\tau\|_{1,p} + |g|_p) \|\beta - \beta_1\|_0 + \\ &+ c(\zeta_0 + L_1)(4/m^2) T_2(\|v\|_{1,p} \|\tau\|_{1,p} + |g|_p) \|\tau - \tau_1\|_0 + \\ &+ c(\zeta_0 + L_1)(2/m)(\|v\|_{1,p} \|\tau\|_{1,p} + |g|_p) S_2(\|\tau - \tau_1\|_0 + \|\beta - \beta_1\|_0) + \\ &+ c(\zeta_0 + L_1)(2/m) T_2 \|\tau\|_{1,p} \|v - v_1\|_0 + \\ &+ \|(\zeta_0 + \beta_1)(m + \tau_1)^{-1} \omega_2(\tau_1, \beta_1)v_1 \cdot \nabla (\tau - \tau_1)\|_{-1}. \end{aligned}$$

We remark that each term on the right hand side of (2) is bounded by the term on the right hand side of (3) that occupies the same relative position. In order to estimate the fourth and the last term on the right hand side of (3) we will use the estimate

$$(4) \quad \|w \cdot \nabla \psi\|_{-1} \leq c(\|w\|_{\infty} + \sum_{i,j} |D_i w_j|_p) \|\psi\|_0.$$

Here w is a vector field and ψ a scalar field in Ω . This inequality is proved by noting that

$$|\int_{\Omega} (w \cdot \nabla \psi) \varphi dx| \leq c(\|\operatorname{div} w\|_p \|\psi\|_0 \|\varphi\|_{2*} + \|w\|_{\infty} \|\psi\|_0 \|\varphi\|_1).$$

By using inequality (4), one easily verifies that the fourth and the last term on the right hand side of (3) are bounded by $c(m+2_1 + \|\tau_1\|_{1,p}) \|v_1\|_{1,p} \|\beta_1\|_0$, and by

$$c(\zeta_0 + 2_1 + \|\beta_1\|_{1,p}) (\frac{2}{m} + \frac{4}{2} \|\tau_1\|_{1,p}) (\tau_2 + S_2 \|\tau_1\|_{1,p} + S_2 \|\beta_1\|_{1,p}) \|v_1\|_{1,p} \|\tau_1\|_0,$$

respectively. By using these estimates together with (3), one gets (3.21).

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ABSTRACT (continued)

solution (u, ρ, ζ) of problem (1.1), (1.3)₁, in a neighborhood of $(0, m, \zeta_0)$; for arbitrarily large data the stationary solution does not exist in general (see section 5). Moreover, we prove that (for barotropic flows) the stationary solution of the Navier-Stokes equations (4.15) is the incompressible limit of the stationary solutions of the compressible Navier-Stokes equations (4.14), as the Mach number becomes small. Finally, in section 5 we will study the equilibrium solutions for system (4.1). For a more detailed explanation see the introduction.

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